

ESD-TDR-65-48

**ESD RECORD COPY**RETURN TO  
SCIENTIFIC & TECHNICAL INFORMATION DIVISION  
(ESTI), BUILDING 1211

COPY NR. \_\_\_\_\_ OF \_\_\_\_\_ COPIES

ESTI PROCESSED☐ DDC TAB ☐ PROJ OFFICER☐ ACCESSION MASTER FILE☐ \_\_\_\_\_

DATE \_\_\_\_\_

ESTI CONTROL NR **AL 45143**

CY NR \_\_\_\_\_ OF \_\_\_\_\_ CYS

**Group Report****1965-8****M. J. Levin****A Method for Power Spectrum  
Parameter Estimation****10 February 1965**Prepared for the Advanced Research Projects Agency  
under Electronic Systems Division Contract AF 19(628)-500 by**Lincoln Laboratory**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts



71150 615 796

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology. This research is a part of Project Vela Uniform, which is sponsored by the U. S. Advanced Research Projects Agency of the Department of Defense; it is supported by ARPA under Air Force Contract AF 19(628)-500 (ARPA Order 512).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
LINCOLN LABORATORY

A METHOD FOR POWER SPECTRUM PARAMETER ESTIMATION

*M. J. LEVIN*

*Group 64*

GROUP REPORT 1965-8

10 FEBRUARY 1965

LEXINGTON

MASSACHUSETTS



## ABSTRACT

An asymptotic analysis is carried out for an approximate method of estimating the parameters of the power spectrum of a zero-mean stationary Gaussian random process from an observed realization of limited duration. Maximum likelihood estimates are obtained with the approximation that the coefficients of the Fourier series expansion of the realization are uncorrelated. This is equivalent to other approximation techniques which assume a periodic covariance function. The dispersion of the estimates is evaluated in terms of a quantity called the differential variance. It is shown that with this quantity as a criterion, the approximate estimates are as good, asymptotically, as the exact maximum likelihood estimates. An approximate expression for the differential variance in terms of the power spectrum is given and it is shown that this expression asymptotically approaches its exact value.

These results follow from a general expression, obtained by means of a converse to the Schwarz inequality, which compares the differential variance of the approximate estimates with that of the maximum likelihood estimates. This expression is evaluated for the power spectrum parameter estimation problem in terms of the covariance matrix of the Fourier coefficients. The asymptotic behavior of these covariances is bounded so that the convergence of the elements of the inverse covariance matrix can be demonstrated. The results on the differential variance of the approximate estimates are then established by matrix methods.

Accepted for the Air Force  
Stanley J. Wisniewski  
Lt Colonel, USAF  
Chief, Lincoln Laboratory Office



## I. INTRODUCTION

This report investigates an approximate method for estimating parameters of the power spectrum of a zero-mean stationary Gaussian random process. Applications and previous work are discussed in references 1-4. The method is based on the approximation that the Fourier coefficients of a realization of long-time duration are uncorrelated. By a result of Root and Pitcher<sup>5</sup> this is equivalent to the approximation that the covariance function of the process is periodic. This latter approximation has been used without quantitative justification by a number of previous authors.<sup>6-9</sup> The burden of the present report is to justify the approximate method by showing that, in a certain asymptotic sense, it provides estimates which are as good as the exact maximum likelihood estimates. This is the first time, to the author's knowledge, that such a quantitative evaluation of the asymptotic behavior of these techniques has appeared. The arguments required have turned out to be somewhat involved, but this is not surprising in view of the general difficulty of treating exactly problems concerning stationary non-white Gaussian processes over a finite time interval. Only the estimation of a single parameter has been dealt with here, but the multi-parameter case can be handled at the cost of still further complexity. The methods can also be applied to the estimation of the parameters of a deterministic signal in additive Gaussian noise and to corresponding detection problems.

In Section II some general results on maximum likelihood estimates, the Cramer-Rao lower bound and the differential variance are discussed. These are applied to power spectrum parameter estimation in Section III. Section IV describes some practical approximations to the estimates and the differential variance. Some results are established in Section V for the convergence of the covariances of the Fourier coefficients and in Section VI for the inverse of the covariance matrix. Finally, Section VII proves that the differential variance of the approximate estimates is as small asymptotically as that of the exact maximum likelihood estimates while Section VIII establishes the validity of the approximation to the differential variance.

## II. MAXIMUM LIKELIHOOD ESTIMATES AND THE DIFFERENTIAL VARIANCE

Some general results of estimation theory,<sup>10</sup> are presented in this Section. Consider an N-dimensional vector random variable  $\underline{x}$  with probability density  $f(\underline{x}; \alpha)$  depending on a parameter  $\alpha$  with true value  $\alpha_0$ . The following discussion assumes that certain general regularity conditions on  $f(\underline{x}; \alpha)$  are fulfilled. For an observation of  $\underline{x}$ , the log likelihood of  $\alpha$  is defined as

$$\Lambda(\underline{x}; \alpha) = \ln f(\underline{x}; \alpha)$$

The maximum likelihood (ML) estimate  $\hat{\alpha}$  is the value of  $\alpha$  which maximizes  $\Lambda(\underline{x}; \alpha)$ . It can sometimes be found explicitly as the root of the likelihood equation

$$\frac{\partial}{\partial \alpha} \Lambda(\underline{x}; \alpha) = 0 \quad (2-1)$$

but usually a linearization in the vicinity of  $\alpha_0$  or a method of successive approximations is required.

The Cramer-Rao lower bound<sup>10</sup> states that for any estimate  $\alpha^*$

$$\text{Var } \alpha^* \geq \frac{[1 + db/d\alpha_0]^2}{E[\Lambda'(\underline{x}; \alpha_0)]^2} \quad (2-2)$$

where

$$b = E\alpha^* - \alpha_0$$

is the bias of  $\alpha^*$ . An alternate expression for the denominator of (2-2) is obtained from the identity

$$E[\Lambda'(\underline{x}; \alpha_0)]^2 = -E\Lambda''(\underline{x}; \alpha_0) \quad (2-3)$$

In this report a prime always denotes differentiation with respect to  $\alpha$  and

$$\Lambda'(\underline{\xi}; \alpha_0) = \left. \frac{\partial}{\partial \alpha} \Lambda(\underline{\xi}; \alpha) \right|_{\alpha = \alpha_0}$$

$$\Lambda''(\underline{\xi}; \alpha_0) = \left. \frac{\partial^2}{\partial \alpha^2} \Lambda(\underline{\xi}; \alpha) \right|_{\alpha = \alpha_0}$$

An unbiased estimate whose variance satisfies (2-2) with the equals sign is said to be efficient. However, efficient estimates exist only in certain cases. ML estimates are not necessarily unbiased or efficient.

Frequently the bias of an estimate under consideration cannot be established so the expression (2-2) is not informative. However, the Cramér-Rao lower bound can still be interpreted in terms of the "sensitivity" introduced by Kelly, Lyons and Root.<sup>11</sup> They consider any statistic  $\alpha^*$  which is a measure of  $\alpha_0$  in the sense that its expectation is a monotonic function of  $\alpha_0$  and define

$$\text{sensitivity} = \frac{\left| \frac{dE\alpha^*}{d\alpha_0} \right|}{\text{standard deviation of } \alpha^* \text{ for } \alpha = \alpha_0}.$$

The reciprocal of the sensitivity is just that small change in  $\alpha_0$  required to change the mean value of  $\alpha^*$  by one standard deviation. It is seen that (2-2) can now be written

$$\left( \frac{1}{\text{sensitivity}} \right)^2 \geq \frac{1}{E[\Lambda'(\underline{\xi}; \alpha_0)]^2} \quad (2-4)$$

Next consider the general class of estimates obtained by maximizing over  $\alpha$  some function  $L(\underline{\xi}; \alpha)$  which depends on  $\underline{\xi}$  and  $\alpha$  (but not, of course, on  $\alpha_0$ ) and has the further property

$$E L'(\underline{\xi}; \alpha_0) = 0 \quad (2-5)$$

The dispersion of the sampling errors can be characterized by a quantity which may be termed the differential variance.

$$S(\alpha^*) = \frac{E[L'(\underline{\xi}; \alpha_0)]^2}{[E L''(\underline{\xi}; \alpha_0)]^2}$$

This quantity can be further interpreted when  $L(\underline{\xi}; \alpha)$  is sufficiently regular so that in the vicinity of the true parameter value  $\alpha_0$ , it can be approximated by the first three terms of the Taylor series expansion

$$L(\underline{\xi}; \alpha) \approx L(\underline{\xi}; \alpha_0) + (\alpha - \alpha_0) L'(\underline{\xi}; \alpha_0) + \frac{(\alpha - \alpha_0)^2}{2} L''(\underline{\xi}; \alpha_0) \quad (2-6)$$

The maximum value of  $L(\underline{\xi}; \alpha)$  over  $\alpha$  occurs where

$$L'(\underline{\xi}; \alpha) = 0 \quad (2-7)$$

so from (2-6)

$$\alpha^* - \alpha_0 \approx - \frac{L'(\underline{\xi}; \alpha_0)}{L''(\underline{\xi}; \alpha_0)} \quad (2-8)$$

The distribution of the sampling error,  $\alpha^* - \alpha_0$ , is approximately the same as that of the expression (2-8). If it is also assumed that the random variations of  $L'(\underline{\xi}; \alpha_0)$  are small compared with  $E L''(\underline{\xi}; \alpha_0)$  then  $L''(\underline{\xi}; \alpha_0)$  can be approximated by its expected value, the random fluctuation of  $L'(\underline{\xi}; \alpha_0)$  about zero may be thought of as a linear noise term displacing the peak of the parabolic approximation to  $L(\underline{\xi}; \alpha)$ , and

$$\text{Var } \alpha^* \approx S(\alpha^*) \quad (2-9)$$

It has been shown by Godambe<sup>12</sup> that if  $S(\alpha^*)$  is taken as a measure of the dispersion of the estimate  $\alpha^*$ , without necessarily referring to the particular interpretation mentioned above, then under general regularity conditions  $S(\alpha^*)$  is minimized when  $L(\underline{\xi}; \alpha) = \Lambda(\underline{\xi}; \alpha)$  and  $\alpha^* = \hat{\alpha}$ . This minimum value is, by (2-3),

$$S(\alpha) = \frac{E[\Lambda'(\underline{\xi}; \alpha_0)]^2}{[E\Lambda''(\underline{\xi}; \alpha_0)]^2} = \frac{1}{E[\Lambda'(\underline{\xi}; \alpha_0)]^2} = \frac{-1}{E\Lambda''(\underline{\xi}; \alpha_0)} \quad (2-10)$$

This is the same as the quantity appearing in the Cramér-Rao lower bound (2-2), but the present result holds with the equals sign for any regular estimate, biased or not.

We now determine an upper bound for  $S(\alpha^*)$  in terms of the difference between  $\Lambda(\underline{\xi}; \alpha)$  and  $L(\underline{\xi}; \alpha)$ . By (2-5)\*

$$\int L'(\underline{x}; \alpha_0) f(\underline{x}; \alpha_0) d\underline{x} = 0 \quad (2-11)$$

The differentiation of (2-11) under the integral sign with respect to  $\alpha_0$  gives

$$\begin{aligned} \int L''(\underline{x}; \alpha_0) f(\underline{x}; \alpha_0) d\underline{x} &= - \int L'(\underline{x}; \alpha_0) f'(\underline{x}; \alpha_0) d\underline{x} \\ &= - \int L'(\underline{x}; \alpha_0) \frac{f'(\underline{x}; \alpha_0)}{f(\underline{x}; \alpha_0)} f(\underline{x}; \alpha_0) d\underline{x} \end{aligned} \quad (2-12)$$

There is a modification of the Schwarz inequality<sup>13</sup> which states that for any non-negative  $\varphi(z)$  and for integrable  $u^2(z)\varphi(z)$  and  $v^2(z)\varphi(z)$

$$\left[ \int u(z) v(z) \varphi(z) dz \right]^2 \geq \left[ 1 - \frac{\int [u(z) - v(z)]^2 \varphi(z) dz}{\int u^2(z) \varphi(z) dz} \right] \int u^2(z) \varphi(z) dz \int v^2(z) \varphi(z) dz \quad (2-13)$$

---

\* Unless otherwise noted, all integrals are N-fold definite integrals, each component of  $\underline{x}$  ranging over  $(-\infty, \infty)$ .

Let us assume the required conditions are satisfied so that (2-13) can be applied to the right-hand side of (2-12) giving

$$\begin{aligned}
 & \left[ \int L'(\underline{x}; \alpha_0) \frac{f'(\underline{x}; \alpha_0)}{f(\underline{x}; \alpha_0)} f(\underline{x}; \alpha_0) d\underline{x} \right]^2 \\
 & \geq \left\{ 1 - \frac{\int \{ L'(\underline{x}; \alpha_0) - [f'(\underline{x}; \alpha_0)/f(\underline{x}; \alpha_0)] \}^2 f(\underline{x}; \alpha_0) d\underline{x}}{\int [L'(\underline{x}; \alpha_0)]^2 f(\underline{x}; \alpha_0) d\underline{x}} \right\} \\
 & \quad \cdot \left\{ \int [L'(\underline{x}; \alpha_0)]^2 f(\underline{x}; \alpha_0) d\underline{x} \int \left[ \frac{f'(\underline{x}; \alpha_0)}{f(\underline{x}; \alpha_0)} \right]^2 f(\underline{x}; \alpha_0) d\underline{x} \right\} \quad (2-14)
 \end{aligned}$$

Noting that

$$\frac{f'(\underline{x}; \alpha_0)}{f(\underline{x}; \alpha_0)} = \Lambda'(\underline{x}; \alpha) \Big|_{\alpha = \alpha_0}$$

and substituting (2-12) into (2-14) we can rearrange the result as

$$\frac{\int [L'(\underline{x}; \alpha_0)]^2 f(\underline{x}; \alpha_0) d\underline{x}}{[\int L''(\underline{x}; \alpha_0) f(\underline{x}; \alpha_0) d\underline{x}]^2} \cdot E[\Lambda'(\underline{\xi}; \alpha_0)]^2 \leq \left\{ 1 - \frac{\int [L'(\underline{x}; \alpha_0) - \Lambda'(\underline{x}; \alpha_0)]^2 f(\underline{x}; \alpha_0) d\underline{x}}{\int [L'(\underline{x}; \alpha_0)]^2 f(\underline{x}; \alpha_0) d\underline{x}} \right\}^{-1} \quad (2-15)$$

The left-hand side of (2-15) is just  $S(\alpha^*)/S(\alpha)$ . The right-hand side consists of a quantity  $(1 - \zeta)^{-1} \geq 1$ . If  $\zeta$  can be shown to converge to zero, then it is established that  $S(\alpha^*)$  converges to its minimum value  $S(\alpha)$ . This is a tractable criterion for establishing the asymptotic behavior of a general class of estimates and is applied in Section VII to the approximate ML power spectrum parameter estimates.

### III. MAXIMUM LIKELIHOOD ESTIMATES FOR POWER SPECTRUM PARAMETERS

The analysis is based on the following assumptions:

a) The random process is stationary, Gaussian, and zero-mean with a double-sided power spectrum  $P(f; \alpha)$  which is a known function of the parameter  $\alpha$  whose value is to be estimated. (If the process consists of a random signal plus an independent noise, then  $P(f; \alpha)$  is the sum of the two individual spectra.)

b) A particular realization  $x(t)$  of the process is observed for  $0 \leq t \leq T$ .

c) The true value of  $\alpha$  is denoted by  $\alpha_0$  and the actual power spectrum is  $\hat{P}(f) = P(f; \alpha_0)$ .  $\alpha_0$  is assumed to lie within a known finite interval

$$\alpha_{\min} < \alpha_0 < \alpha_{\max}.$$

(d) For any  $\alpha$  within this interval, only values of  $P(f; \alpha)$  within some fixed finite range

$$0 < f_1 < |f| < f_2 < \infty$$

depend on  $\alpha$ . Since  $f_1$  and  $f_2$  may be anywhere outside of the range of dependence for convenience, we take

$$f_1 = N_1/T \quad f_2 = N_2/T$$

where  $N_1$  and  $N_2$  are integers. (This assumption means that  $P(f; \alpha)$  has some constant shape outside of  $f_1 < |f| < f_2$ . The assumption  $f_2 < \infty$  allows a finite-dimensional formulation of the problem and is meaningful in the usual practical situation where frequencies above some finite limit are not observable. The assumption  $f_1 > 0$  is only for the convenience of eliminating zero frequency which enters unsymmetrically into the formulation.)

e)  $P(f; \alpha)$  obeys certain regularity conditions which will be detailed in Section V.

The Karhunen-Loeve expansion which is frequently employed in the analysis of random processes is not well adapted to spectral parameter estimation problems since it is in terms of an uncorrelated set of eigenfunctions which, except in the

simplest cases, change in a complicated way as the parameters vary. Instead, the likelihood function is here expressed in terms of the Fourier coefficients

$$\gamma_n = \frac{1}{\sqrt{T}} \int_0^T x(t) \exp [-i2\pi n f_0 t] dt \quad (3-1)$$

where

$$f_0 = 1/T$$

The coefficients for  $N_1 \leq n \leq N_2$  are taken as the "observable coordinates"<sup>13</sup> of the process. This is plausible since under general conditions, for almost every realization,  $x(t)$  can be represented by the limit-in-the-mean of its Fourier series expansion.<sup>5</sup> This choice of the observable coordinates could be further justified but that will not be attempted here.

The real and imaginary parts of  $\gamma_n$  each have a Gaussian distribution with zero-mean and the entire set of

$$N = N_2 - N_1 + 1 = (f_2 - f_1) T + 1 \leq f_2 T \quad (3-2)$$

coefficients has an N-variate complex Gaussian distribution. It can be shown<sup>5</sup> that

$$r_{nm} = E \gamma_n \bar{\gamma}_m = \frac{1}{\pi^2 T} \int df \Phi(f) \frac{\sin^2 \pi f T}{(f - n/T)(f - m/T)} \quad (3-3)$$

where  $\bar{\gamma}_m$  is the complex conjugate of  $\gamma_m$ . (3-3) is equivalent to a result in Reference 5 except for a correction of a factor of two. The boundedness of  $\Phi(f)$  assures that the  $r_{nm}$  exist and the normalization in (3-1) is chosen so that  $E|\gamma_n|^2$  has a finite limit as  $T \rightarrow \infty$ .

With  $\underline{R}$  as the matrix having elements  $r_{nm}$  and  $\underline{\gamma}$  as the vector with elements  $\gamma_n$ , the log likelihood of the parameter  $\alpha$  is

$$\Lambda(\underline{Y}; \alpha) = - \{ \ln \pi^N |R| + \underline{Y}^T \underline{R}^{-1} \underline{Y} \} \quad (3-4)$$

which is maximized over  $\alpha$  to determine  $\hat{\alpha}$ . It is shown in Appendix A that  $\underline{R}$  is non-singular if

$$\Phi(f) > 0 \quad \text{for all } f$$

Next, let  $\underline{D}$  be the matrix with elements

$$d_{nm} = \frac{\partial}{\partial \alpha} r_{nm} = \frac{1}{\pi^2 T} \int df P'(f; \alpha) \frac{\sin^2 \pi f T}{(f - n/T)(f - m/T)} \quad (3-5)$$

and assume all  $d_{nm}$  exist. Using matrix relations given by Bodewig<sup>15</sup> we have

$$\frac{\partial}{\partial \alpha} \ln |R| = \text{tr } \underline{R}^{-1} \underline{D} \quad (3-6)$$

$$\frac{\partial}{\partial \alpha} \underline{R}^{-1} = -\underline{R}^{-1} \underline{D} \underline{R}^{-1} \quad (3-7)$$

and thus

$$\Lambda'(\underline{Y}; \alpha) = -\{ \text{tr } \underline{R}^{-1} \underline{D} - \underline{Y}^T \underline{R}^{-1} \underline{D} \underline{R}^{-1} \underline{Y} \} \quad (3-8)$$

(In certain cases an exact expression for  $\hat{\alpha}$  can be found by using (3-8) in (2-1).) The fourth product moment for complex Gaussian variates<sup>16</sup> is

$$E(\underline{Y}^T \underline{A} \underline{Y})(\underline{Y}^T \underline{B} \underline{Y}) = \text{tr } \underline{A} \underline{R} \text{tr } \underline{B} \underline{R} + \text{tr } \underline{A} \underline{R} \underline{B} \underline{R} \quad (3-9)$$

and therefore

$$E[\Lambda'(\underline{Y}; \alpha_0)]^2 = \text{tr } \underline{R}^{-1} \underline{D} \underline{R}^{-1} \underline{D} \quad (3-10)$$

which provides the Cramér-Rao lower bound and the value of  $S(\hat{\alpha})$ .

#### IV. APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATES FOR POWER SPECTRUM PARAMETERS

Unfortunately, the exact expression (3-4) for  $\Lambda(\underline{y}; \alpha)$  is too complicated to permit calculations in most cases. However, it is easily seen that

$$\lim_{T \rightarrow \infty} r_{nm} = \begin{cases} \hat{\Phi}(nf_0) & n = m \\ 0 & n \neq m \end{cases} \quad (4-1)$$

A great simplification occurs if the approximation is made that for large  $T$

$$r_{nm} = \begin{cases} \hat{\Phi}(nf_0) & n = m \\ 0 & n \neq m \end{cases} \quad (4-2)$$

Then  $\underline{R}$  becomes diagonal and we take

$$\Lambda(\underline{y}; \alpha) \approx L(\underline{y}; \alpha) = - \sum_{n=N_1}^{N_2} \left[ \ln \pi P(nf_0; \alpha) + \frac{|\gamma_n|^2}{P(nf_0; \alpha)} \right] \quad (4-3)$$

Thus the likelihood is approximated in terms of the elements of the periodogram  $|\gamma_n|^2$ .  $\alpha^*$  will henceforth denote the approximate ML estimates obtained by maximizing  $L(\underline{y}; \alpha)$ .

If  $P(f; \alpha)$  is a slowly varying function, then the summation (4-3) can be approximated by an integral. The individual  $|\gamma_n|^2$  need not be measured since their values can be smoothed and replaced in this integral by a continuous function  $\hat{\Phi}^*(f)$  which is equivalent to an approximately unbiased overall spectral estimate such as given by the Blackman-Tukey method<sup>17</sup> or by a power spectrum analyzer. For the integral to be a good approximation all the  $|\gamma_n|^2$  must be effectively included and the resolution must be sufficient so that the detailed structure of the spectrum is not obscured. With (4-2),  $|\gamma_n|^2$  is replaced by  $\hat{\Phi}^*(nf_0)$  in (4-3) and

$$L(\underline{y}; \alpha) \approx -T \int_{f_1}^{f_2} df \left[ \log \pi P(f; \alpha) + \frac{\hat{\Phi}^*(f)}{P(f; \alpha)} \right] \quad (4-4)$$

where the integral is multiplied by  $T$ , the reciprocal of the spacing of the terms in the summation.

It is now readily seen that

$$L'(\underline{y}; \alpha) = - \sum_n \left[ \frac{P'(nf_0; \alpha)}{P(nf_0; \alpha)} - \frac{|\gamma_n|^2 P'(nf_0; \alpha)}{P^2(nf_0; \alpha)} \right] \quad (4-5)$$

and that

$$E L'(\underline{y}; \alpha_0) = 0 \quad (4-6)$$

which satisfies (2-5). A second differentiation and the substitution of (4-2) give

$$- E L''(\underline{y}; \alpha_0) = \sum_n \left[ \frac{P'(nf_0; \alpha_0)}{P(nf_0; \alpha_0)} \right]^2 \approx T \int_{f_1}^{f_2} df \left[ \frac{P'(f; \alpha_0)}{P(f; \alpha_0)} \right]^2 \quad (4-7)$$

With the discussion in Appendix B and (3-2), it can readily be established that  $E L''(\underline{y}; \alpha_0)$  is of the order of  $T$  while the absolute value of the error in approximating the sum by the integral is bounded by a constant. By some further calculations of this sort the approximation introduced just before (2-9) can also be justified for this case.

We will take  $- E L''(\underline{y}; \alpha_0)$  as a convenient approximation to  $- E \Lambda''(\underline{y}; \alpha_0)$ , the quantity entering into (2-2) and (2-10). It should be noted that

$$E [L'(\underline{y}; \alpha_0)]^2 \neq - E L''(\underline{y}; \alpha_0)$$

because the  $\gamma_n$  are not, in general, uncorrelated.

## V. CONVERGENCE OF THE COVARIANCES OF THE FOURIER COEFFICIENTS

For our purposes it is necessary to show that for some  $T_0 > 0$  and  $T > T_0$  the covariances  $r_{nm}$  converge uniformly and with sufficient rapidity to their asymptotic values for all  $n, m$  satisfying

$$\begin{aligned} T f_1 &\leq n \leq T f_2 \\ T f_1 &\leq m \leq T f_2 \end{aligned} \tag{5-1}$$

This can be established with the assumptions on  $P(f; \alpha)$  given in Section III plus the one below.

(e) For all  $f$  and all  $\alpha$  within  $\alpha_{\min} < \alpha < \alpha_{\max}$  there exist finite constants  $P_{\max}$ ,  $Q_{\max}$ ,  $P'_{\max}$ ,  $Q'_{\max}$ ,  $P_{\min}$  such that

$$\begin{aligned} P(f; \alpha) &\leq P_{\max} \\ \left| \frac{\partial P(f; \alpha)}{\partial f} \right| &\leq Q_{\max} \\ \left| \frac{\partial P(f; \alpha)}{\partial \alpha} \right| &\leq P'_{\max} \\ \left| \frac{\partial^2}{\partial f \partial \alpha} P(f; \alpha) \right| &\leq Q'_{\max} \\ P(f; \alpha) &\geq P_{\min} > 0 \end{aligned}$$

and all indicated derivatives exist and are continuous.

The variances  $r_{nn}$  are considered first. From (3-3)

$$\begin{aligned} r_{nn} &= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \, \Phi(f) \frac{\sin^2 \pi f T}{(f - n/T)^2} \\ &= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \, \Phi(f + n/T) \frac{\sin^2 \pi f T}{f^2} \\ &= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \, [\Phi(n/T) + \Phi(f + n/T) - \Phi(n/T)] \frac{\sin^2 \pi f T}{f^2} \end{aligned} \tag{5-2}$$

Since

$$\frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \frac{\sin^2 \pi f T}{f^2} = 1 \quad (5-3)$$

then

$$\begin{aligned} r_{nn} - \phi(n/T) &= \epsilon_{Rn} \\ &= \frac{1}{\pi^2 T} \left\{ \int_{-2f_2}^{2f_2} + \int_{-\infty}^{-2f_2} + \int_{2f_2}^{\infty} \right\} \left\{ df [\phi(f + n/T) - \phi(n/T)] \frac{\sin^2 \pi f T}{f^2} \right\} \end{aligned} \quad (5-4)$$

By the mean value theorem for the derivative

$$|\phi(f + n/T) - \phi(n/T)| \leq |f| Q_{\max} \quad (5-5)$$

so

$$\begin{aligned} |\epsilon_{Rn}| &\leq \frac{1}{\pi^2 T} \left[ \int_{-2f_2}^{2f_2} df Q_{\max} \frac{\sin^2 \pi f T}{|f|} + \left\{ \int_{-\infty}^{-2f_2} + \int_{2f_2}^{\infty} \right\} \right. \\ &\quad \left. \left\{ df P_{\max} \frac{\sin^2 \pi f T}{f^2} \right\} \right] \end{aligned} \quad (5-6)$$

With the substitution

$$u = \pi f T$$

(5-6) becomes

$$\begin{aligned} |\epsilon_{Rn}| &\leq \frac{1}{\pi^2 T} \left[ Q_{\max} \int_{-2\pi T f_2}^{2\pi T f_2} du \frac{\sin^2 u}{|u|} \right. \\ &\quad \left. + 2P_{\max} \int_{2f_2}^{\infty} df \frac{1}{f^2} \right] \end{aligned} \quad (\text{Con't.})$$

$$\begin{aligned}
&= \frac{1}{\pi^2 T} \left[ 2Q_{\max} \left( \int_0^{\pi} du \frac{\sin u}{u} + \int_{\pi}^{2\pi T f_2} \frac{du}{u} \right) + \frac{P_{\max}}{f_2} \right] \\
&= \frac{1}{\pi^2 T} \left[ 2Q_{\max} (\pi + \ln 2 T f_2) + \frac{P_{\max}}{f_2} \right] \quad (5-7)
\end{aligned}$$

which is of the order of  $(\ln T)/T$ .

Next, for  $m \neq n$ ,

$$\begin{aligned}
r_{nm} &= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \, \Phi(f + n/T) \frac{\sin^2 \pi f T}{f(f - \frac{m-n}{T})} \\
&= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \, [\Phi(f + n/T) - \Phi(n/T)] \frac{\sin^2 \pi f T}{f(f - \frac{m-n}{T})} \quad (5-8)
\end{aligned}$$

since

$$\int_{-\infty}^{\infty} df \frac{\sin^2 \pi f T}{f(f - \frac{m-n}{T})} = 0 \quad \text{for } m \neq n \quad (5-9)$$

Proceeding as before, we have

$$\begin{aligned}
r_{nm} &= \frac{1}{\pi^2 T} \left[ \left\{ \int_{-2f_2}^{2f_2} + \int_{-\infty}^{-2f_2} + \int_{2f_2}^{\infty} \right\} df \, [\Phi(f + n/T) - \Phi(n/T)] \frac{\sin^2 \pi f T}{f(f - \frac{m-n}{T})} \right] \\
&= \frac{1}{\pi^2 T} \left[ \int_{-2f_2}^{2f_2} df \, Q_{\max} \frac{\sin^2 \pi f T}{|f - \frac{m-n}{T}|} + P_{\max} \left( \int_{-\infty}^{-2f_2} df \frac{1}{(f+f_2)^2} + \int_{2f_2}^{\infty} df \frac{1}{(f-f_2)^2} \right) \right] \\
&= \frac{1}{\pi^2 T} \left[ Q_{\max} \int_{-3f_2}^{3f_2} df \frac{\sin^2 \pi f T}{|f|} + \frac{2P_{\max}}{f_2} \right] \\
&= \frac{1}{\pi^2 T} \left[ 2Q_{\max} (\pi + \ln 3 T f_2) + \frac{2P_{\max}}{f_2} \right] \quad (5-10)
\end{aligned}$$

Thus, by (5-7) and (5-10), for  $T$  larger than any specified  $T_0$ , we can write

$$|r_{nm} - \delta_{nm} \phi(n/T)| \leq K_R \frac{\ln T}{T} \quad (5-11)$$

where  $\delta_{nm}$  is the Kronecker delta and  $K_R$  is a finite constant which depends on  $T_0$  but is independent of  $T$ ,  $\alpha$ ,  $n$  and  $m$ . This rate of convergence is sufficiently rapid to allow proof of the final results.

It is also necessary to establish the asymptotic behavior of

$$\begin{aligned} d_{nm} &= \left. \frac{\partial}{\partial \alpha} r_{nm} \right|_{\alpha = \alpha_0} \\ &= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df P'(f; \alpha_0) \frac{\sin^2 \pi f T}{(f - n/T)(f - m/T)} \end{aligned}$$

First consider

$$\begin{aligned} d_{nn} &= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df [P'(n/T; \alpha_0) + P'(f + n/T; \alpha_0) - P'(n/T; \alpha_0)] \frac{\sin^2 \pi f T}{f^2} \\ &= P'(n/T; \alpha_0) + \epsilon_{Dn} \end{aligned} \quad (5-12)$$

Then

$$|\epsilon_{Dn}| \leq \frac{1}{\pi^2 T} \left\{ \int_{-2f_2}^{2f_1} + \int_{-\infty}^{-2f_1} + \int_{2f_2}^{\infty} \right\} \left\{ df |P'(f + n/T; \alpha_0) - P'(n/T; \alpha_0)| \frac{\sin^2 \pi f T}{f^2} \right\} \quad (5-13)$$

By the assumptions on  $P'(f; \alpha)$  and the mean value theorem for the derivative,

$$|P'(f + n/T; \alpha_0) - P'(n/T; \alpha_0)| \leq |f| Q'_{\max} \quad (5-14)$$

so a development analogous to that given previously shows

$$|\epsilon_{Dn}| \leq \frac{1}{\pi^2 T} \left[ 2Q'_{\max} (\pi + \ln 2Tf_2) + \frac{2P'_{\max}}{f_2} \right] \quad (5-15)$$

An analogous development also holds for  $m \neq n$ , so for  $T > T_0$

$$|d_{nm} - \delta_{nm} P'(n/T; \alpha_0)| \leq K_D \frac{\ln T}{T} \quad (5-16)$$

where  $K_D$  is a finite constant independent of  $T$ ,  $\alpha$ ,  $n$  and  $m$ .

## VI. CONVERGENCE OF THE INVERSE OF THE COVARIANCE MATRIX OF THE FOURIER COEFFICIENTS

Let  $\lambda_{\min}(\underline{R})$  be the smallest eigenvalue of  $\underline{R}$  and let  $(\underline{R})_{nm}^{-1}$  be the  $(n, m)$  element of  $\underline{R}^{-1}$ . It has been shown elsewhere<sup>18</sup> that

$$\frac{1}{r_{nn}} \leq (\underline{R}^{-1})_{nn} \leq \frac{1}{r_{nn} - (\sum_{i=1}^{N_1} r_{in}^2) / \lambda_{\min}(\underline{R})} \quad (6-1)$$

where  $\sum'$  indicates that the term in the summation corresponding to  $i = n$  is omitted. In Section V it was established that for  $T > T_0$

$$|r_{in}| \leq K_R \frac{\ln T}{T} \quad i \neq n \quad (6-2)$$

so with (3-2)

$$\left| \sum_{i=1}^{N_1} r_{in}^2 \right| \leq f_2 K_R^2 \frac{(\ln T)^2}{T} \quad (6-3)$$

By Appendix A

$$\lambda_{\min}(\underline{R}) \geq P_{\min} \quad (A-5)$$

so

$$\frac{1}{r_{nn}} \leq (\underline{R}^{-1})_{nn} \leq \frac{1}{r_{nn} - C (\ln T)^2 / T} \quad (6-4)$$

where

$$C = \frac{f_2 K_R^2}{P_{\min}} > 0 \quad (6-5)$$

Thus, with (5-11), for  $T > T_0$

$$\frac{1}{\hat{\phi}(n/T) + K_R \frac{\ln T}{T}} \leq (\underline{R}^{-1})_{nn} \leq \frac{1}{\hat{\phi}(n/T) - K_R \frac{\ln T}{T} - C \frac{(\ln T)^2}{T}} \quad (6-6)$$

For any  $\varphi > 0$  and  $|\epsilon| \leq \varphi/2$

$$\frac{1}{\varphi} - \frac{\epsilon}{\varphi} \leq \frac{1}{\varphi + \epsilon} \leq \frac{1}{\varphi} + \frac{2|\epsilon|}{\varphi^2}$$

and therefore

$$\frac{1}{\hat{\phi}(n/T)} - \frac{K_R}{p_{\min}^2} \frac{\ln T}{T} \leq (\underline{R}^{-1})_{nn} \leq \frac{1}{\hat{\phi}(n/T)} + \frac{2}{p_{\min}^2} \left[ K_R \frac{\ln T}{T} + C \frac{(\ln T)^2}{T} \right] \quad (6-7)$$

It can also be shown that<sup>18</sup>

$$|(\underline{R}^{-1})_{nm}| \leq \frac{1}{\lambda_{\min}^2(\underline{R})} [ |r_{nm}| + \left( \sum_{i=1}^{N_i} r_{in}^2 \right) / \lambda_{\min}(\underline{R}) ] \quad (6-8)$$

With (A-5), (6-2) and (6-3), it follows that

$$|(\underline{R}^{-1})_{nm}| \leq \frac{1}{p_{\min}^2} [ K_R \ln T/T + C (\ln T)^2/T ] \quad (6-9)$$

Combining this with (6-7) gives for  $T > T_0$

$$\begin{aligned} |(\underline{R}^{-1})_{nm} - \delta_{nm} \hat{\phi}(n/T)| &\leq \frac{2}{p_{\min}^2} \left[ \frac{K_R \ln T}{T} + \frac{C (\ln T)^2}{T} \right] \\ &\leq C_R \frac{(\ln T)^2}{T} \end{aligned} \quad (6-10)$$

where  $C_R$  is a finite constant which depends upon  $T_0$  but not upon  $T$ ,  $\alpha$ ,  $n$ , or  $m$ . Thus as  $T$  becomes large, the off-diagonal elements of  $\underline{R}^{-1}$  converge to zero as  $(\ln T)^2/T$ .

## VII. ASYMPTOTIC BEHAVIOR OF THE APPROXIMATE ESTIMATES

It will now be established that

$$\lim_{T \rightarrow \infty} \frac{S(\alpha^*)}{S(\hat{\alpha})} = 1 \quad (7-1)$$

so that under the criterion of smallness of the differential variance  $\alpha^*$  is asymptotically as good as  $\hat{\alpha}$ . This is accomplished by showing that

$$\lim_{T \rightarrow \infty} \zeta = 0 \quad (7-2)$$

where

$$\begin{aligned} \zeta &= \frac{\int [L'(\underline{x}; \alpha) - \Lambda'(\underline{x}; \alpha)]^2 f(\underline{x}; \alpha) d\underline{x}}{\int [L'(\underline{x}; \alpha)]^2 f(\underline{x}; \alpha) d\underline{x}} \\ &= \frac{\zeta_{\text{num}}}{\zeta_{\text{den}}} \end{aligned}$$

appears in (2-15).

By (4-7) and Appendix B,  $\zeta_{\text{den}}$  is of the order of  $T$ . The development which follows is devoted to establishing that  $\zeta_{\text{num}}$  is of the order of  $(\ln T)^4$  so that asymptotically  $\zeta$  is of the order of  $(\ln T)^4/T$  and therefore satisfies (7-2).

Let  $\underline{R}$  be the diagonal matrix whose diagonal elements are  $R_{nn} = P(n/T; \alpha_0)$  and whose off-diagonal elements are zero. Let  $\Delta$  have diagonal elements  $\Delta_{nn} = P'(n/T; \alpha_0)$  and off-diagonal elements zero. From (3-8), (4-5) and (3-9)

$$\begin{aligned} \zeta_{\text{num}} &= E[\text{tr } \underline{R}^{-1} \underline{D} - \text{tr } \underline{R}^{-1} \underline{\Delta} - \underline{Y}^T \underline{R}^{-1} \underline{D} \underline{R}^{-1} \underline{Y} + \underline{Y}^T \underline{R}^{-1} \underline{\Delta} \underline{R}^{-1} \underline{Y}]^2 \\ &= [\text{tr } (\underline{R}^{-1} \underline{D} - \underline{R}^{-1} \underline{\Delta})]^2 - 2 [\text{tr } (\underline{R}^{-1} \underline{D} - \underline{R}^{-1} \underline{\Delta})] [\text{tr } (\underline{R}^{-1} \underline{D} - \underline{R}^{-1} \underline{\Delta} \underline{R}^{-1} \underline{R})] \\ &\quad + [\text{tr } (\underline{R}^{-1} \underline{D} - \underline{R}^{-1} \underline{\Delta} \underline{R}^{-1} \underline{R})]^2 + 2 \text{tr } (\underline{R}^{-1} \underline{D} - \underline{R}^{-1} \underline{\Delta} \underline{R}^{-1} \underline{R})^2 \end{aligned}$$

Con't.

$$\begin{aligned}
&= [\text{tr} (\underline{\mathcal{R}}^{-1} \underline{\Delta} \underline{\mathcal{R}}^{-1} \underline{R} - \underline{\mathcal{R}}^{-1} \underline{\Delta})]^2 + \text{tr} (\underline{\mathcal{R}}^{-1} \underline{D} - \underline{\mathcal{R}}^{-1} \underline{\Delta} \underline{\mathcal{R}}^{-1} \underline{R})^2 \\
&= \zeta_{\text{num } 1} + \zeta_{\text{num } 2}
\end{aligned} \tag{7-3}$$

By (5-11)

$$\begin{aligned}
\zeta_{\text{num } 1} &\leq \left[ \sum_n \frac{P'(n/T; \alpha_0)}{P^2(n/T; \alpha_0)} (r_{nn} - P(n/T; \alpha_0)) \right]^2 \\
&\leq \left[ \frac{f_2 P'_{\max} K_R \ln T}{P_{\min}^2} \right]^2
\end{aligned} \tag{7-4}$$

$\zeta_{\text{num } 2}$  is bounded by observing that for any square matrix  $\underline{A}$

$$\text{tr} \underline{A}^2 \leq \text{tr} \underline{A} \underline{A}^T = \sum_{nm} a_{nm}^2 \tag{7-5}$$

So if

$$|b_{nm}| \geq |a_{nm}| \quad \text{for all } n, m \tag{7-6}$$

then

$$\text{tr} \underline{A}^2 \leq \sum_{n, m} b_{nm}^2 \tag{7-7}$$

Thus we can write, by (5-11), (5-16) and (5-10),

$$\begin{aligned}
\zeta_{\text{num } 2} &= \text{tr} [(\underline{\mathcal{R}}^{-1} + \underline{\mu}_1) (\underline{\Delta} + \underline{\mu}_2) - \underline{\mathcal{R}}^{-1} \underline{\Delta} \underline{\mathcal{R}}^{-1} (\underline{\mathcal{R}} + \underline{\mu}_3)]^2 \\
&= \text{tr} [\underline{\mathcal{R}}^{-1} \underline{\mu}_2 + \underline{\mu}_1 \underline{\Delta} + \underline{\mu}_1 \underline{\mu}_2 - \underline{\mathcal{R}}^{-1} \underline{\Delta} \underline{\mathcal{R}}^{-1} \underline{\mu}_3]^2 \\
&\leq \text{tr} [\underline{\mathcal{R}}^{-1} \underline{\eta}_2 + \underline{\eta}_1 \underline{\Delta} + \underline{\eta}_1 \underline{\eta}_2 + \underline{\mathcal{R}}^{-1} \underline{\Delta} \underline{\mathcal{R}}^{-1} \underline{\eta}_3]^2
\end{aligned} \tag{7-8}$$

where  $\underline{\Delta}^+$  has elements which are the absolute values of those of  $\underline{\Delta}$ , all elements of  $\underline{\eta}_1$ , are equal to  $C_R \frac{(\ln T)^2}{T}$ , all elements of  $\underline{\eta}_2$  are equal to  $K_D \frac{\ln T}{T}$  and all elements of  $\underline{\eta}_3$  are equal to  $K_R \frac{\ln T}{T}$ .

Then

$$\zeta_{\text{num } 2} \leq (f_2 T)^2 \left[ \frac{K_D \ln T}{P_{\min} T} + P'_{\max} C_R \frac{(\ln T)^2}{T} + C_R K_D \frac{(\ln T)^3}{T^2} + \frac{P'_{\max} K_R \ln T}{P_{\min}^2 T} \right]^2 \quad (7-9)$$

From (7-4) and (7-9),  $\zeta_{\text{num}}$  is of the order of  $(\ln T)^4$  for  $T > T_0$ . Since  $\zeta_{\text{den}}$  is of the order of  $T$ , we conclude that  $\zeta$  is of the order of  $\frac{(\ln T)^4}{T}$  or less which establishes (7-2) and hence the desired result.

# VIII. CONVERGENCE OF THE APPROXIMATION TO THE DIFFERENTIAL VARIANCE

It has been shown (3-10) that

$$S(\hat{\alpha}) = \{ \text{tr } \underline{R}^{-1} \underline{D} \underline{R}^{-1} \underline{D} \}^{-1} \quad (8-1)$$

A convenient approximation (4-7) for  $S(\hat{\alpha})$  is

$$\begin{aligned} \{ -E L''(\underline{Y}; \alpha_0) \}^{-1} &= \left\{ \sum_n \frac{P'(n/T; \alpha_0)}{P(n/T; \alpha_0)} \right\}^{-1} \\ &= \{ \text{tr } \underline{R}^{-1} \underline{\Delta} \underline{R}^{-1} \underline{\Delta} \}^{-1} \end{aligned} \quad (8-2)$$

It was shown that for  $T > T_0$  this latter expression is of the order of  $1/T$ . It will now be established that as  $T \rightarrow \infty$  this expression approaches  $S(\hat{\alpha})$  and in fact

$$\lim_{T \rightarrow \infty} \frac{\{ -E L''(\underline{Y}; \alpha_0) \}^{-1}}{S(\hat{\alpha})} = 1 \quad (8-3)$$

With the notation introduced in Section VII, (3-10) and (4-7)

$$\begin{aligned} \frac{-E L''(\underline{Y}; \alpha_0)}{-E L''(\underline{Y}; \alpha_0)} &= \frac{\text{tr } (\underline{R}^{-1} \underline{D})^2}{\text{tr } (\underline{R}^{-1} \underline{\Delta})^2} \\ &= \frac{\text{tr } [(\underline{R}^{-1} + \underline{\mu}_1) (\underline{\Delta} + \underline{\mu}_2)]^2}{\text{tr } (\underline{R}^{-1} \underline{\Delta})^2} \end{aligned} \quad (8-4)$$

so

$$\left| \frac{-E L''(\underline{Y}; \alpha_0)}{-E L''(\underline{Y}; \alpha_0)} - 1 \right| \leq \frac{\text{tr } (\underline{R}^{-1} \underline{\eta}_2 + \underline{\eta}_1 \underline{\Delta} + \underline{\eta}_1 \underline{\eta}_2)^2}{\text{tr } (\underline{R}^{-1} \underline{\Delta})^2} \quad (8-5)$$

By the same type of argument employed in the previous Section it can be shown that the right hand side of (8-5) is of the order of  $(\ln T)^4/T$  so that (8-3) is established and therefore the validity of the approximation of  $S(\hat{\alpha})$  by  $\{ -E L''(\underline{Y}; \alpha_0) \}^{-1}$  is confirmed.

# APPENDIX A. EXTREME EIGENVALUES OF THE COVARIANCE MATRIX OF THE FOURIER COEFFICIENTS

For any vector  $\underline{x}$ ,

$$\lambda_{\max}(\underline{R}) = \max_{\underline{x}} \frac{\underline{x}^T \underline{R} \underline{x}}{\underline{x}^T \underline{x}} \quad (\text{A-1})$$

Now

$$\begin{aligned} \underline{x}^T \underline{R} \underline{x} &= \sum_{n,m=N_1}^{N_2} x_n x_m \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \hat{\Phi}(f) \frac{\sin^2 \pi f T}{(f - n/T)(f - m/T)} \\ &= \frac{1}{\pi^2 T} \int_{-\infty}^{\infty} df \hat{\Phi}(f) \left[ \sum_n x_n \frac{\sin \pi f T}{(f - n/T)} \right]^2 \end{aligned} \quad (\text{A-2})$$

Since the integrand of (A-2) is non-negative

$$\begin{aligned} \underline{x}^T \underline{R} \underline{x} &\leq \frac{P_{\max}}{\pi^2 T} \int_{-\infty}^{\infty} df \left[ \sum_n x_n \frac{\sin \pi f T}{(f - n/T)} \right]^2 \\ &\leq \frac{P_{\max}}{\pi^2 T} \sum_{n,m} x_n x_m \int_{-\infty}^{\infty} df \frac{\sin^2 \pi f T}{(f - n/T)(f - m/T)} \\ &\leq P_{\max} \sum_n x_n^2 \end{aligned} \quad (\text{A-3})$$

and

$$\lambda_{\max}(\underline{R}) \leq P_{\max} \quad (\text{A-4})$$

Similarly,

$$\lambda_{\min}(\underline{R}) \geq P_{\min} \quad (\text{A-5})$$

Hence, if  $P_{\min} > 0$ ,  $\underline{R}$  is positive definite.

## APPENDIX B. THE APPROXIMATION OF SUMS BY INTEGRALS

Assume that  $u(x)$  and  $\frac{du(x)}{dx}$  are continuous (and therefore bounded) for  $-\infty < a \leq x \leq b < \infty$  and that

$$I = \int_a^b u(x) dx$$

exists. Then

$$I = \sum_{n=0}^{N-1} I_n \quad (B-1)$$

where

$$I_n = \int_{a+n\Delta}^{a+(n+1)\Delta} u(x) dx$$

and

$$N\Delta = (b - a)$$

By the mean value theorem for the derivative

$$|u(x) - u(n\Delta)| \leq \Delta u'_{\max} \quad \text{for } n\Delta \leq x \leq (n+1)\Delta \quad (B-2)$$

where

$$u'_{\max} = \max_x \left| \frac{du(x)}{dx} \right| \quad \text{for } a \leq x \leq b$$

By the first law of the mean

$$I_n = \Delta u(\xi) \quad (B-3)$$

for some  $\xi$ , such that

$$a + n\Delta < \xi < a + (n+1)\Delta \quad (\text{B-4})$$

Hence

$$\begin{aligned} |I_n - \Delta u(n\Delta)| &\leq \max_{a + n\Delta \leq x \leq a + (n+1)\Delta} \Delta |u(x) - u(n\Delta)| \\ &\leq \Delta^2 u'_{\max} \end{aligned} \quad (\text{B-5})$$

by (B-2). Therefore

$$|I - \Delta \sum_{n=0}^{N-1} u(n\Delta)| \leq \Delta^2 u'_{\max} = \Delta(b-a) u'_{\max} \quad (\text{B-6})$$

and the difference between the integral and the finite approximating sum is of the order of  $\Delta$ . On the other hand

$$\left| \sum_{n=0}^{N-1} u(n\Delta) - \frac{1}{\Delta} \int_a^b u(x) dx \right| \leq (b-a) u'_{\max} \quad (\text{B-7})$$

The right-hand side of (B-7) is a constant independent of  $N$ . Since

$$\frac{1}{\Delta} = \frac{N}{(b-a)}$$

the left-hand side of (B-7) is of the order of  $N$  for  $N > N_0 > 0$ .

## REFERENCES

1. Levin, M. J. , "Power Spectrum Parameter Estimation," IEEE Trans. on Information Theory, IT-11; January 1964.
2. Scheppe, F. C. , "Evaluation of Likelihood Functions for Gaussian Signals," IEEE Trans. on Information Theory, IT-11; January 1964.
3. Hofstetter, E. M. , "Some Results on the Stochastic Signal Parameter Estimation Problem," ICMCI Conference, Tokyo, Japan; September, 1964.
4. Sakrison, D. J. , "Efficient Recursive Estimation of the Parameters of a Covariance Function," Internal Technical Memorandum M-76, Electronics Research Laboratory, University of California, Berkeley; July, 1964.
5. Root, W. L. and T. S. Pitcher, "On the Fourier Series Expansion of Random Functions," Annals Math. Stat. , 26; June, 1955 (313-318).
6. Whittle, P. , Hypothesis Testing in Time Series Analysis (Almqvist and Wiksells, Uppsala; 1951).
7. Whittle, P. , Appendix in A Study in the Analysis of Stationary Time Series, by H. Wold, 2nd edition (Stockholm; 1953).
8. Whittle, P. , "Estimation and Information in Stationary Time Series," Ark. Mat. , 2; 1953 (423-434).
9. Good, I. J. , "Weighted Covariance for Detecting the Direction of a Gaussian Source," Proc. of Symposium on Time Series Analysis, Brown University, June 11-14, 1962, edited by M. Rosenblatt (John Wiley and Sons, 1963) (447-470).
10. Cramer, H. , Mathematical Methods of Statistics (Princeton University Press; 1951).
11. Kelly, E. J. , D. H. Lyons and W. L. Root, "The Sensitivity of Radiometric Measurements," J. of the S. I. A. M. , 11; June 1963 (235-257).
12. Godambe, V. P. , "An Optimum Property of Regular Maximum Likelihood Estimation," Annals of Math. Stat. , 31; December, 1960 (1208-1211).
13. Levin, M. J. , "On the Schwarz Inequality," Correspondence Section, Proc. IEEE, 53; January 1965 (107-108).

14. Grenander, U. , "Stochastic Processes and Statistical Inference," Ark. Mat. , 1; 1950 (195-277).
15. Bodewig, E. , Matrix Calculus (Interscience Publishers, New York; 1956).
16. Bello, P. , "Some Results on the Problem of Discriminating Between Two Gaussian Processes," IRE Trans. on Information Theory, IT-7; October, 1961 (224-233).
17. Blackman, R. B. and J. W. Tukey, The Measurement of Power Spectra from the Point of View of Communications Engineering, (Dover Publications, New York; 1959).
18. Levin, M. J. , "Bounds on the Inverse of a Positive Definite Symmetric Matrix," Group Report 1964-67, Lincoln Laboratory, M. I. T. ; November 1964.

## DOCUMENT CONTROL DATA - DD 1473

1. ORIGINATING ACTIVITY  Lincoln Laboratory, M.I.T.		2a. REPORT CLASSIFICATION <u>Unclassified</u>	
		2b. DOWNGRADING GROUP	
3. REPORT TITLE  A Method for Power Spectrum Parameter Estimation			
4. TYPE OF REPORT AND INCLUSIVE DATES Group Report			
5. AUTHOR(S) (Last name first)  Levin, M. Joseph			
6. REPORT DATE 10 February 1965		7a. NO. OF PAGES 34	7b. NO. OF REFS. 18
8a. CONTRACT NO. AF 19(628)-500		9a. ORIGINATOR'S REPORT NO. Group Report 1965-8	
8b. ORDER NO. ARPA Order 512		9b. OTHER REPORT NO(S). ESD-TDR-65-48	
10. AVAILABILITY OR LIMITATION NOTICES			
11. SUPPLEMENTARY NOTES		12. SPONSORING ACTIVITY Advanced Research Projects Agency	
13. ABSTRACT <p>An asymptotic analysis is carried out for an approximate method of estimating the parameters of the power spectrum of a zero-mean stationary Gaussian random process from an observed realization of limited duration. Maximum likelihood estimates are obtained with the approximation that the coefficients of the Fourier series expansion of the realization are uncorrelated. The dispersion of estimates is evaluated in terms of a differential variance. With this quantity, the approximation estimates are shown to be as good asymptotically as the exact maximum likelihood estimates. An approximate expression for the differential variance in terms of the power spectrum is given, which is shown to asymptotically approach its exact value.</p> <p>These results follow from a general expression obtained in terms of a converse to the Schwarz inequality, which compares the differential variances of the approximate estimates and the maximum likelihood estimates. This expression is evaluated for the power spectrum parameter estimation problem in terms of the covariance matrix of the Fourier coefficients. The asymptotic behavior of these covariances is bounded to demonstrate the convergence of the inverse covariance matrix elements. The results on the differential variance of approximate estimates are established by matrix methods.</p>			
14. KEY WORDS  power spectra  Gaussian processes			





